CS 221

Tuesday 8 November 2011

Agenda

1. Announcements
2. Review: Solving Equations (Text 6.1-6.3)
3. Root-finding with Excel ("Goal Seek", Text 6.5)
4. Example Root-finding Problems
5. The Fixed-point root-finding method
6. Error estimation in numerical methods
7. Review: Matrix Mathematics (Text Ch. 7)
8. Matrix Operations in MATLAB & Excel (Ch. 7)

1. Announcements

- Problem Set 4 out tonight, due Wed 16 Nov
  - Root-finding problems
- Next Quiz: in Class, two weeks from today: Tuesday, 22 November

2. Review: Solving Equations

- Step one: get the equation into the form:

$$f(x) = 0$$

- Step two: determine what is the form of $f(x)$
- Step three: solve accordingly
  - Linear: use algebra
  - Nonlinear - polynomial
    - MATLAB: use roots() – preferred: gives all roots
    - Excel: use "Goal Seek"
  - Nonlinear - general
    - MATLAB: use fzero()
    - Excel: use "Goal Seek"

Determine the Form of *f(x)*

- Does *f(x)* contain powers of *x* higher than 1, or transcendental functions like cos, sin, exp, log, etc.?
  - no: linear
  - yes: see below
- Does *f(x)* contain noninteger powers of *x*, or transcendentals?
  - no: nonlinear/polynomial
  - yes: nonlinear/general

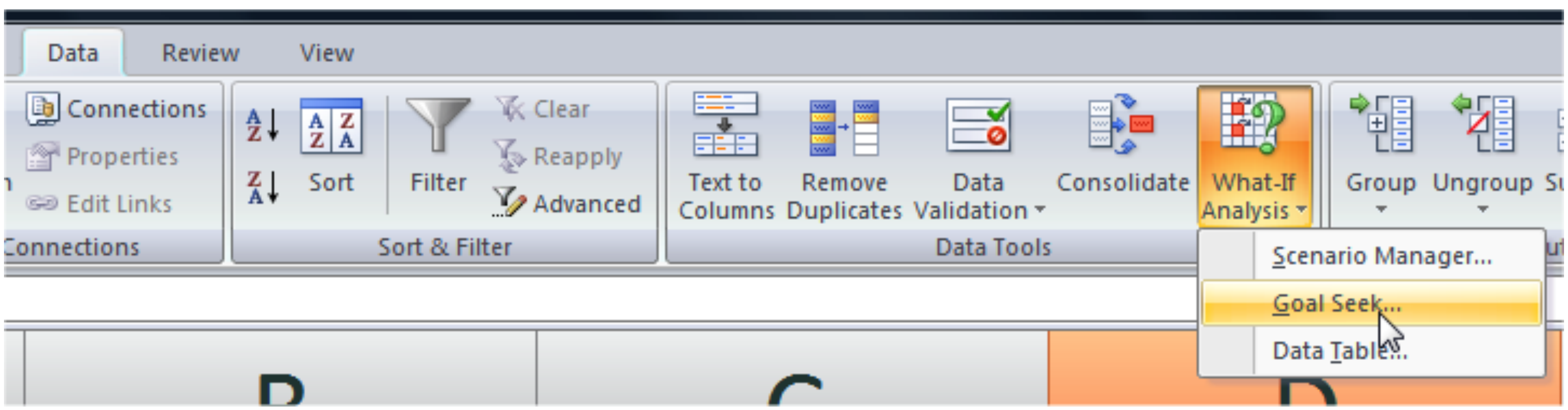
3 Root-Finding with Excel
(Text Section 6.5)

- Simple tool: "Goal Seek" capability
- Tells Excel to vary the value of a cell containing the independent variable until value of the cell containing the function (dependent variable) is equal to a given value
- Steps:
  1. Identify one cell to hold the value of the independent variable (call it x)
  2. Fill in another cell with the formula for f(x)
  3. Put an initial estimate of the root in the first cell
  4. Select the "goal seek" function and set the target value to 0.

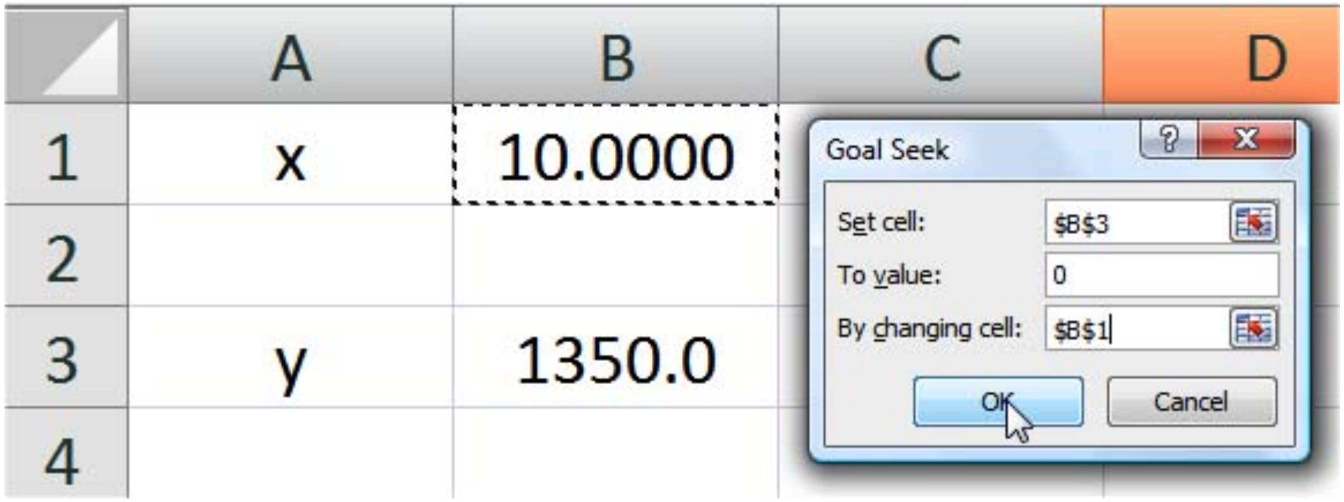
Goal Seek Example

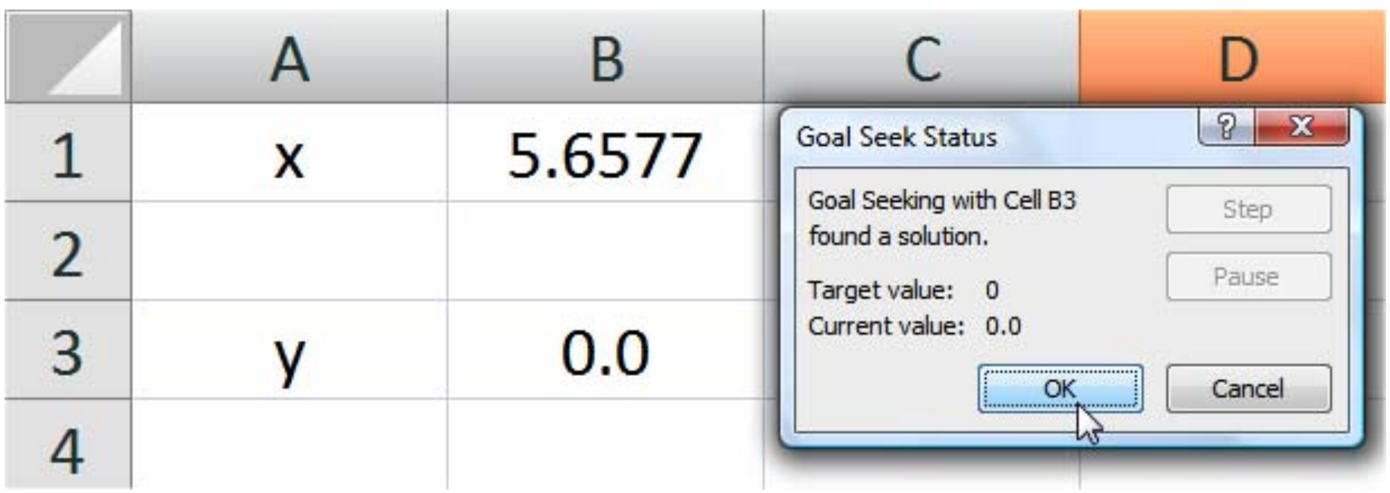
$$y = 3x^3 - 15x^2 - 20x + 50$$

	A	B
1	x	10.0000
2		
3	y	1350.0



Goal Seek Example





Use the MATLAB “roots()” function to solve polynomials.

- Because f(x) is a polynomial in this example, is easily solved in this way
- Create a vector for the coefficients of the polynomial: c = [3, -15, -20, 50];
- Pass it to roots():   roots(c)
- Result is a column vector containing <u>all</u> roots of the polynomial – real and complex

4. Example Problems

Thermistors are resistors (electrical components) whose resistance varies with temperature, according to the Steinhart-Hart equation:

$$1/(T + 273.15) = C_1 + C_2 \ln(R) + C_3 \ln^3(R)$$

What resistance (in ohms) corresponds to a temperature of 15° C, if C_1 = 1.1E-3, C_2 = 2.3E-4, and C_3=8.2E-8?

Get the Equation into the Right Form

We are solving for R, so we want f(R) = 0:

1/(T + 273.15) = 1.1E-3 + 2.3E-4*log(R) + 8.2E-8*log(R)^3

1.1E-3 + 2.3E-4*log(R) + 8.2E-8*log(R)^3 – (1/(T+273.15)) = 0

Use plotting tools (fplot) to determine the neighborhood of the root.

Example

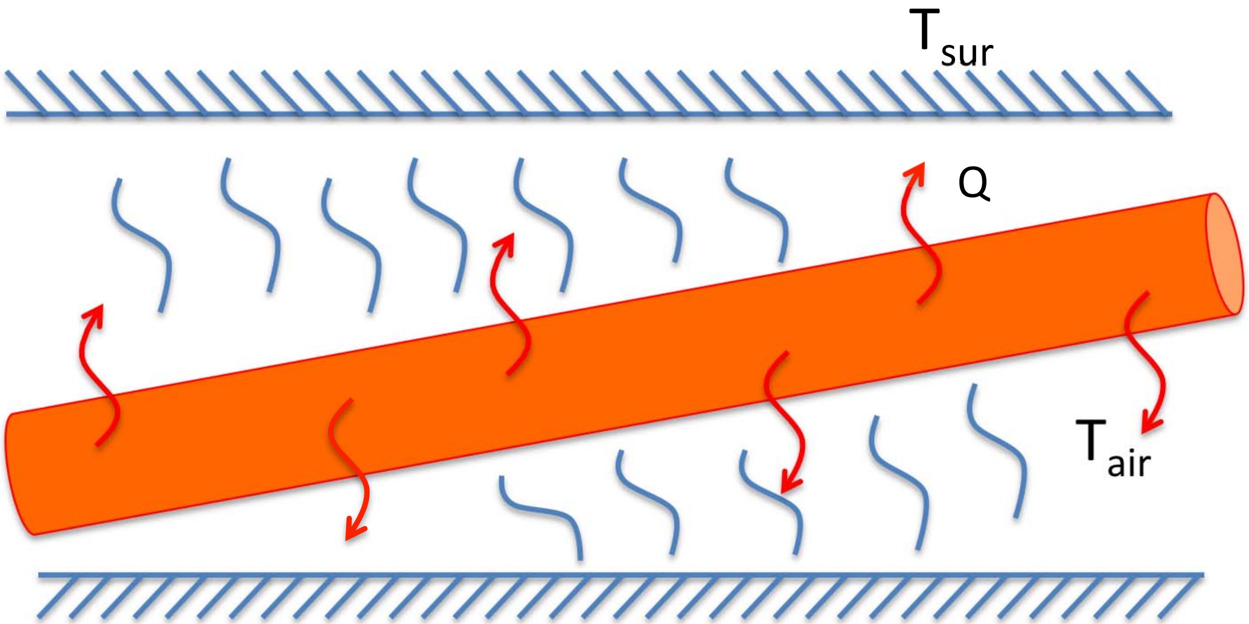
- You are designing a water tank for a village in Africa. The tank is spherical, with a radius R meters. The volume of water it holds if filled to height h is given by:

  $$V = \pi h^2(3R-h)/3$$

- To what height must the tank be filled to hold 30 m^3 of water?

Another Example Problem

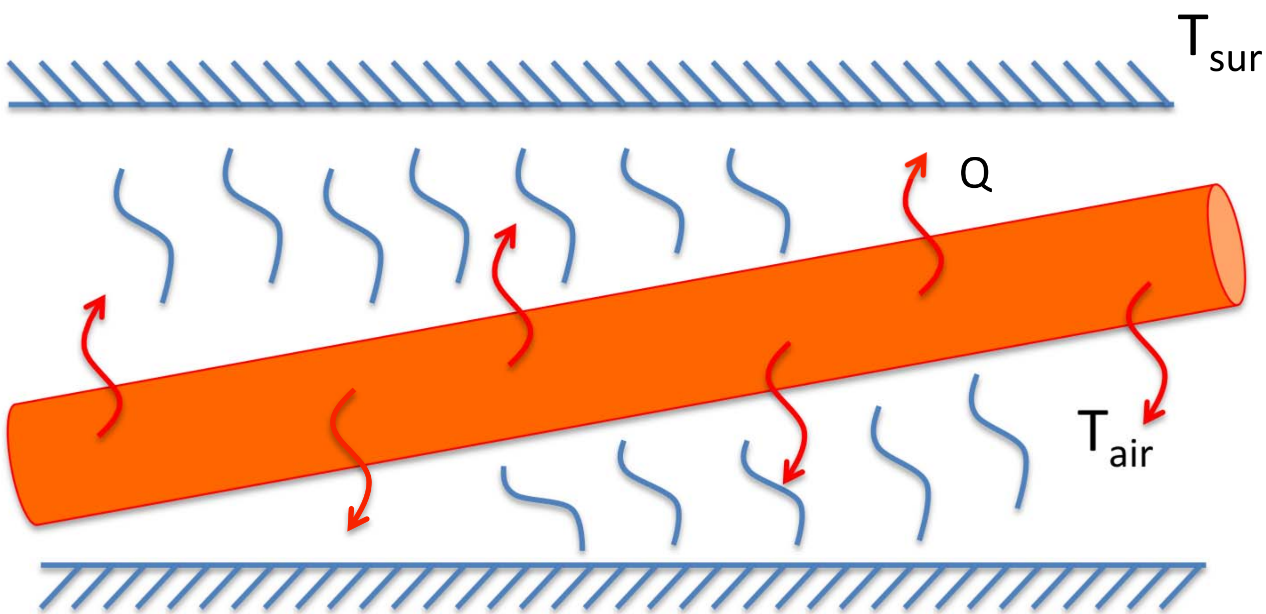
- Consider a steam pipe of length L=25m and diameter d=0.1m. The pipe loses heat to the ambient air and surrounding surfaces by convection and radiation.



Another Example Problem

- The relationship between the surface temperature of the pipe, T_S, and the total flow of heat per unit time, Q, is:

$$Q = \pi d L[h(T_S - T_{air}) + \varepsilon \sigma_{SB}(T_S^4 - T^4_{sur})]$$



Another Example Problem

- Where:
  - $h = 10\ W/m^2/K$
  - $\varepsilon = 0.8$
  - $\sigma_{SB} = 5.67 \times 10^{-8}$ = Stefan-Bolzmann Constant
  - $T_{air} = T_{sur} = 298K$

- If Q = 18405 W, what is the surface temperature (T_S) of the pipe?

$$\pi dL[h(T_S - T_{air}) + \varepsilon\sigma_{SB}(T_S^4 - T^4_{sur})] - Q = 0$$

5. The Fixed-Point Method of Finding Roots (note: not in your text)

- A value x is a fixed-point of function $g(x)$ iff

$$g(x) = x$$

- Two basic concepts:
  1. Rewriting an equation $f(x) = 0$ into fixed-point form: $g(x) = x$
  2. Iterative method of finding a fixed point:

```
while x_i ≠ x_{i+1}
    x_{i+1} = g(x_i)
end
```

Actually, we will use $|x_i - x_{i+1}| > \varepsilon$

Rewriting Equations

- We need to rewrite so that $f(x) = 0$ if $g(x) = x$
- If f(x) contains an x term, rearrange to get it on one side:

  Example: $x^2 - 2x - 3 = 0$

  becomes $(x^2 - 3)/2 = x$
- Or, you can always just add x to both sides:

  $f(x) = 0$ if $f(x) + x = x$, so take $g(x) = f(x) + x$
  - Example: $x^2 - 2x - 3 = 0$

    $x^2 - 2x - 3 + x = x$

    $x^2 - x - 3 = x$

Rewriting Equations

Do I hear you saying “Wait a minute...”

In that example, $f(x) = x^2 - 2x - 3 = 0$,

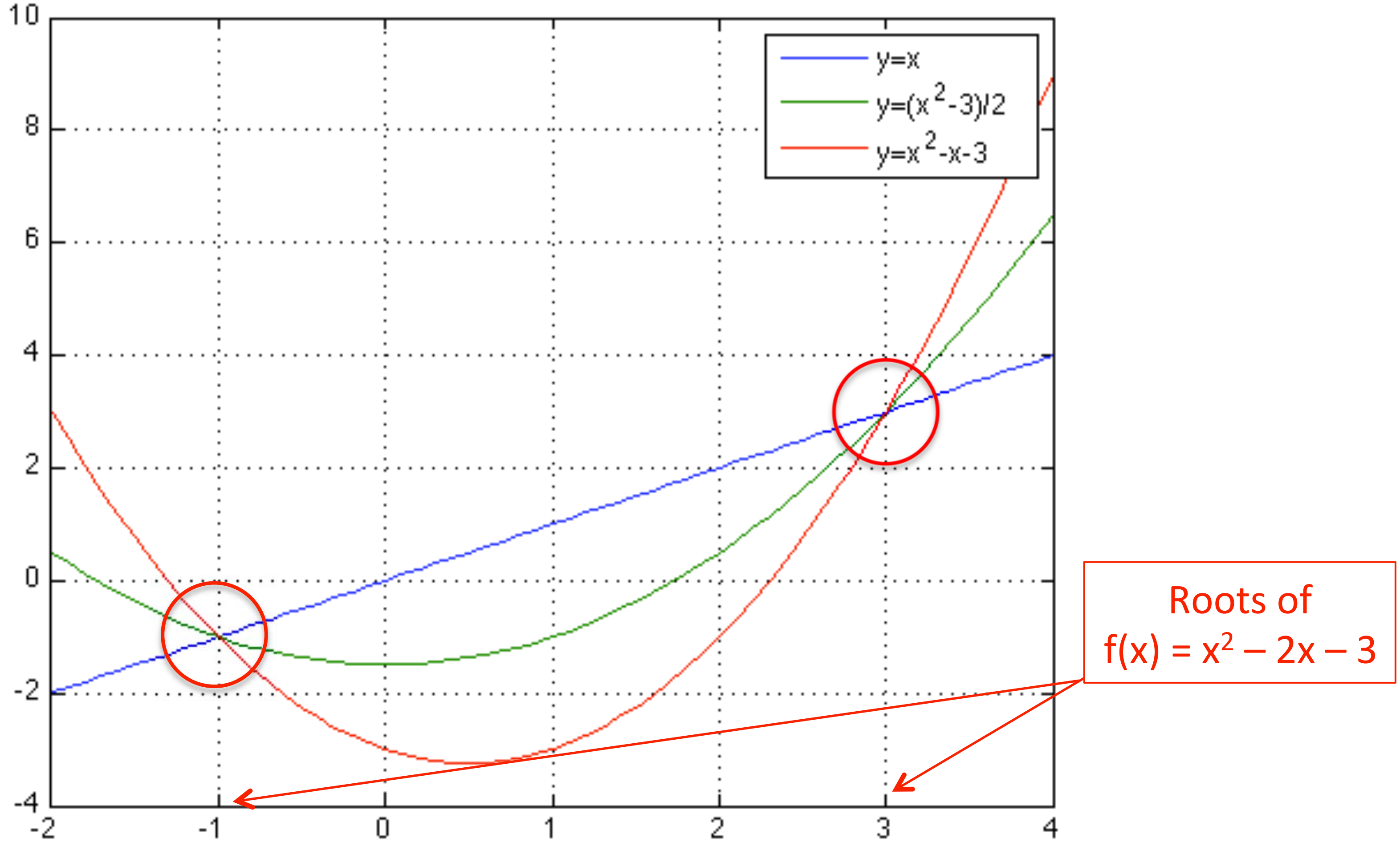
the two methods yield two <u>different</u> g(x)’s!

$$g(x) = (x^2 - 3)/2$$

and $$g(x) = x^2 - x - 3$$

What gives??!!

Answer: Both g(x)'s work



Rewriting Equations

- Goal: find x such that $f(x) = 0$.
- We are transforming this into the problem of finding x such that $x = g(x)$
  - In such a way that $f(x) = 0$ for any such x
- There may be more than one $g(x)$ for which this relationship holds!
- Graphically: we are looking for the intersection between the curves $y = x$ and $y = g(x)$

The Iterative Method is Simple!

- Main loop:

  set x_0 = an estimate of the root

  While error estimate is too big:

  set $x_{i+1} = g(x_i)$;

  update error estimate;

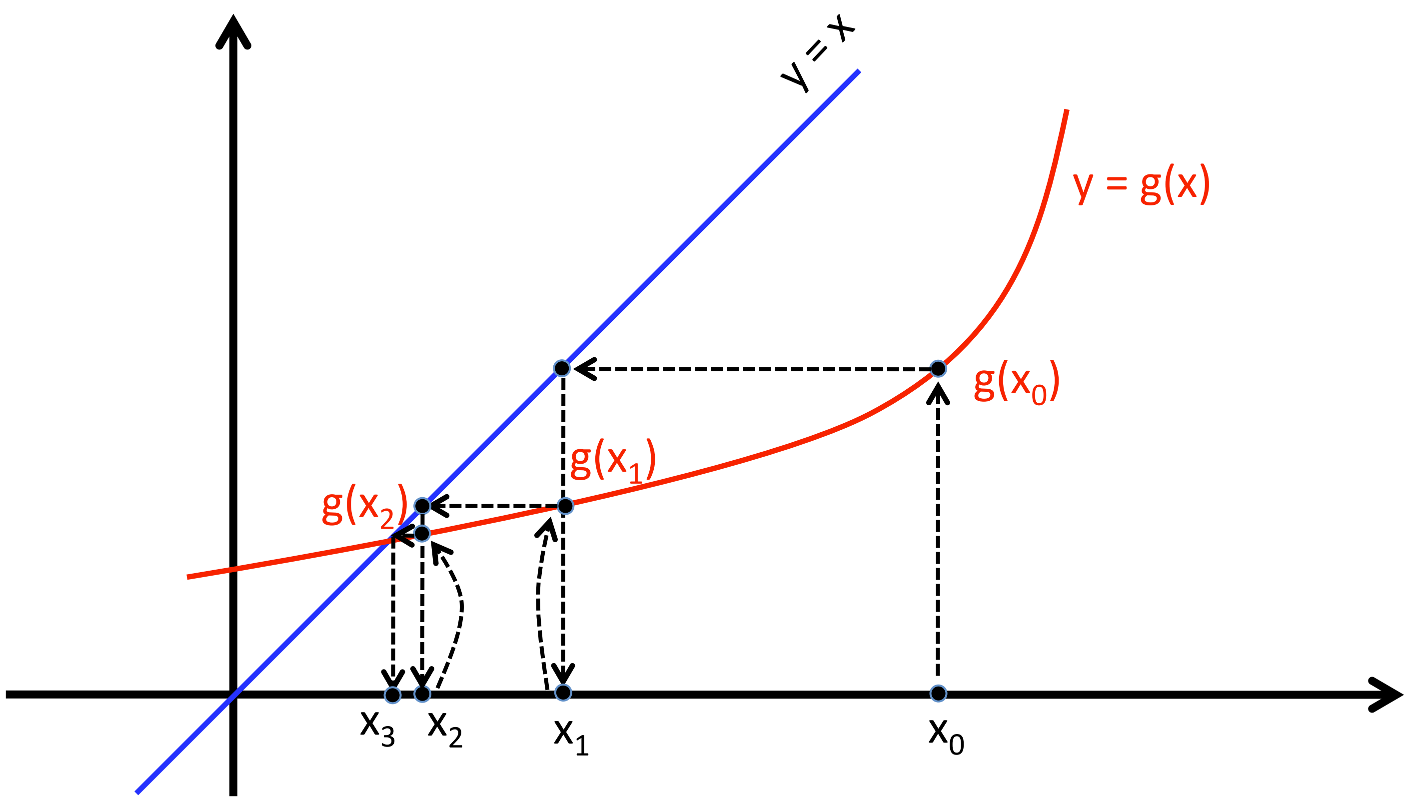
  end

- Approximate relative error estimate:

  $\varepsilon_a = |(x_{i+1} - x_i)/x_{i+1}|$

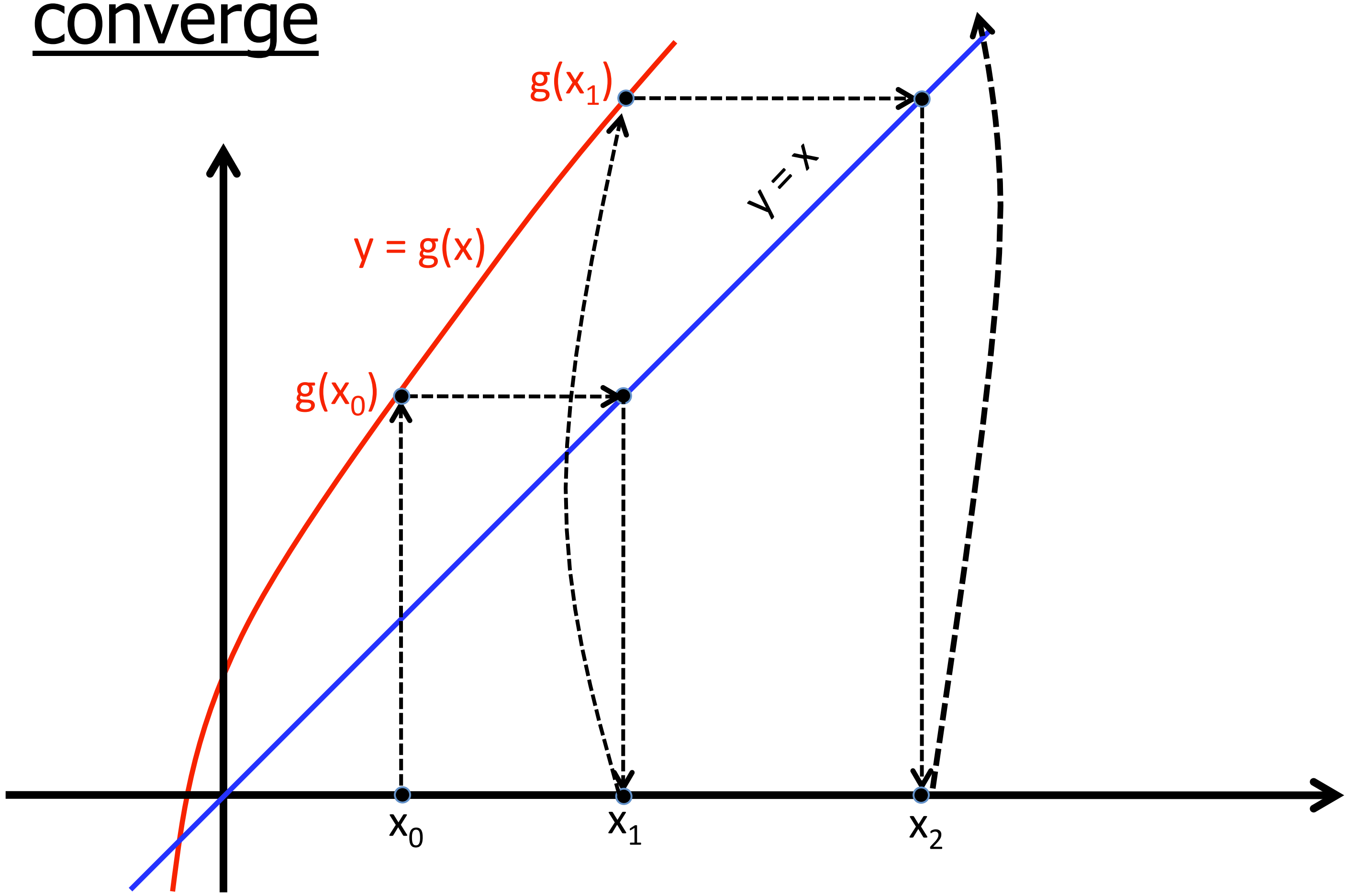
Convergence and Divergence

- Graphical depiction of the fixed-point method



Convergence and Divergence

- The fixed-point method is <u>not guaranteed to converge</u>



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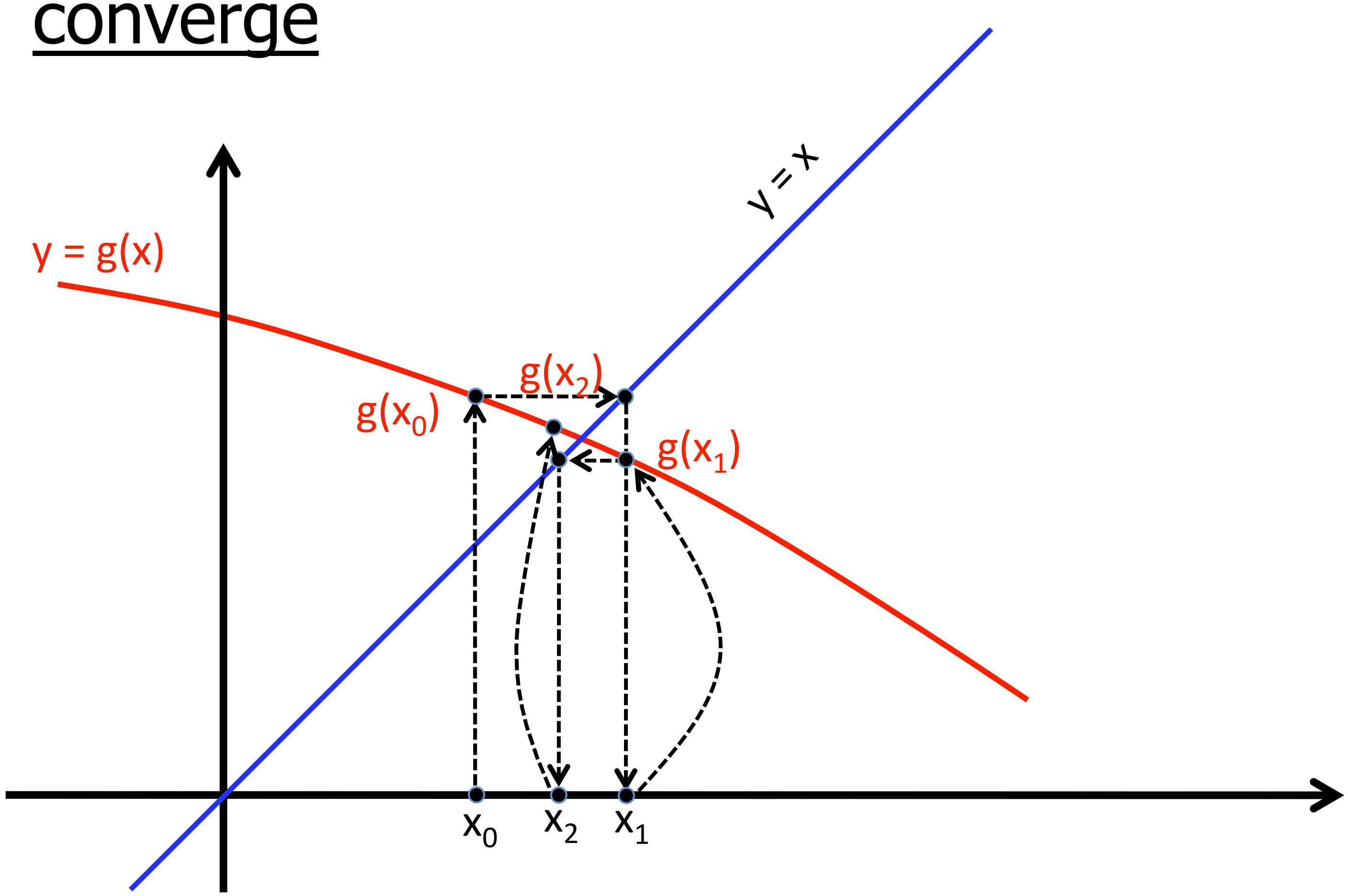

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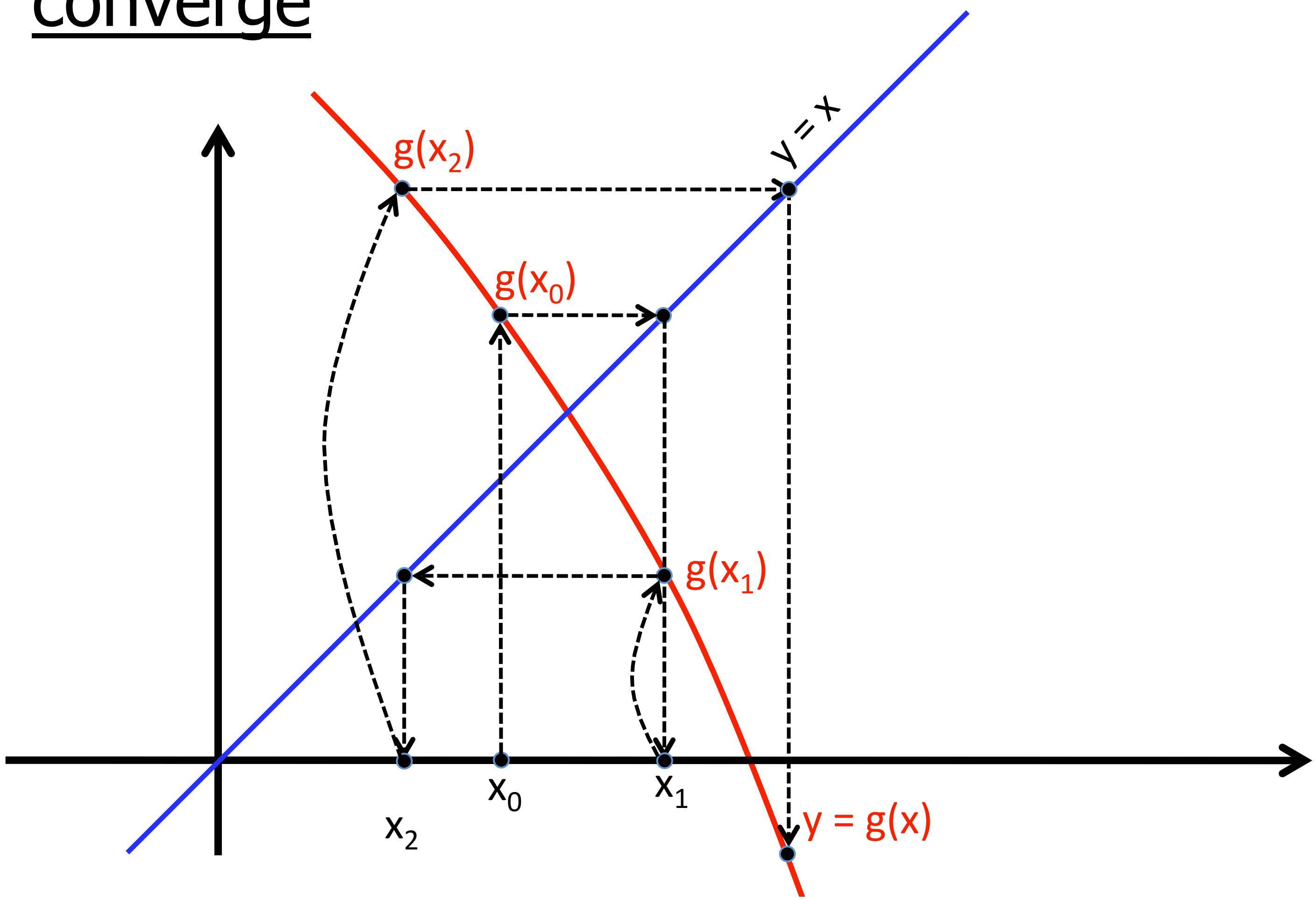

Convergence of Fixed-Point Iteration

- Convergence is determined by the magnitude of the slope:

$$\left| \frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} \right|$$

If |slope| < 1 then $g(x_{i+1})$ will be closer to the fixed point than $g(x_i)$ (the error decreases)

Else $g(x_{i+1})$ does not approach the fixed point

(error increases or stays the same)

6. Error Estimation in Numerical Methods

- Numerical methods yield approximate results
- As engineers, we need to understand the sources of error in our results
  - Round-off or truncation errors
    - E.g., using 3.14 for pi
  - Computational methods
    - Order of operations can make a difference
    - Rewrite formulas to avoid operations whose operands have very different magnitudes
  - Inputs (e.g., measurements)
  - Approximations of functions (Mont Blanc tunnel)

Definitions

- True value = approximation + error

  or:

  E_t = true value – approximation ("true error")

  Note: requires knowledge of true value!

  $\varepsilon_t = E_t$/true value ("true relative error", usually given as a percentage)

  Example:

  - measure a city block: 9999 cm
    - True value: 10000 cm
  - measure a pencil: 9 cm
    - True value: 10 cm
  - Error is 1 cm in both cases!
  - Relative errors are very different: 0.01% vs 10%

Error Estimation

- When we use numerical methods, we usually don't know the "true" value!
  - We are computing an approximation, and would like to know how close we are to the real value
  - Typically we have to estimate the error as well!
- Best-effort approach:

  ε_a = approximate error/approximate value ("approximate relative error")
- For iterative methods:

  ε_a = (current estimate – previous estimate)/current estimate
- A bound ε_s on the error may be specified as part of the problem

  Iterate until $|\varepsilon_a| < \varepsilon_s$

7. Review: Matrix Mathematics