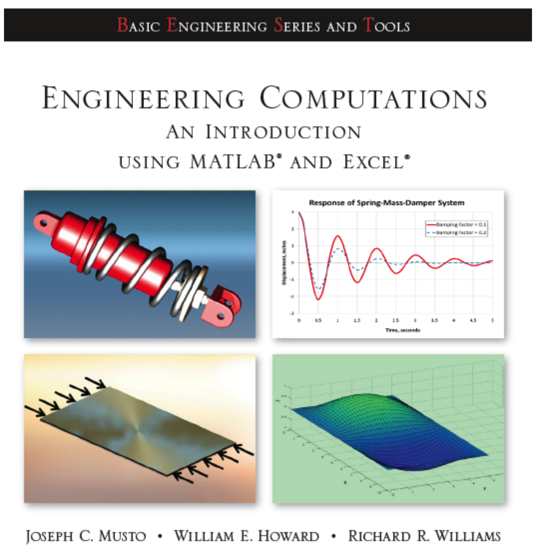
Matrix Operations



Chapter 7
Matrix Mathematics

Matrix Mathematics

- Matrices are very useful in engineering calculations. For example, matrices are used to:
  - Efficiently store a large number of values (as we have done with arrays in MATLAB)
  - Solve systems of linear simultaneous equations
  - Transform quantities from one coordinate system to another
- Several mathematical operations involving matrices are important

Review: Properties of Matrices

- A matrix is a one-or two dimensional array
- A quantity is usually designated as a matrix by bold face type: **A**
- The elements of a matrix are shown in square brackets:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 6 \\ 7 & -1 & 3 \\ 0 & 3 & 0.5 \end{bmatrix}$$

Review: Properties of Matrices

- The dimension (size) of a matrix is defined by the number of rows and number of columns
- Examples:

3 × 3:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 6 \\ 7 & -1 & 3 \\ 0 & 3 & 0.5 \end{bmatrix}$$

2×4:

$$\mathbf{B} = \begin{bmatrix} 3 & 5 & -2 & 1 \\ 0 & 4 & 10 & 3 \end{bmatrix}$$

Review: Properties of Matrices

- An element of a matrix is sometimes written in lower case, with its row number and column number as su[illegible]

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- In MATLAB, an element is designated by the matrix name with the row and column numbers in parentheses: A(1,2)

Matrix Operations

- Matrix Addition
- Multiplication of a Matrix by a Scalar
- Matrix Multiplication
- Matrix Transposition
- Finding the Determinant of a Matrix
- Matrix Inversion

Matrix Addition

- Matrices (or vectors) must be the same size in order to add
- To add two matrices, add the individual elements:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

- Matrix addition is commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

and associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Multiplication of a Matrix by a Scalar

- To multiple a matrix by a scalar, multiply each element by the scalar:

$$k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

- MATLAB often uses this fact to simplify the display of matrices with very large (or very small) values:

$$\begin{bmatrix} 29{,}000{,}000 & 1{,}600{,}000 \\ 1{,}600{,}000 & 12{,}000{,}000 \end{bmatrix} = \begin{bmatrix} 29 & 1.6 \\ 1.6 & 12 \end{bmatrix} \times 10^6$$

Multiplication of Matrices

- To multiply two matrices together, the matrices must have compatible sizes:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

  This multiplication is possible only if the number of columns in **A** is the same as the number of rows in **B**

- The resulting matrix **C** will have the same number of rows as **A** and the same number of columns as **B**

Multiplication of Matrices

- Consider these matrices:

$$\mathbf{A} = \begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

- Can we find this product? $\mathbf{C} = \mathbf{A} \times \mathbf{B}$

Yes, 3 columns of A = 3 rows of B

- What will be the size of **C**?

2 X 2: 2 rows in A, 2 columns in B

Multiplication of Matrices

- Easy way to remember rules for multiplication:

These values must match

$$(2 \text{ X } 3)(3 \text{ X } 2)$$

Size of Product Matrix

Multiplication of Matrices

- Element *ij* of the product matrix is computed by multiplying each element of row *i* of the first matrix by the corresponding element of column *j* of the second matrix, and summing the results
- This may be easier to see with an example

Example – Matrix Multiplication

- Find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$

$$\mathbf{A} = \begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

- We know that matrix C will be 2 × 2
- Element c_{11} is found by summing the products of elements in row 1 of **A** and column 1 of **B**:

$$c_{11} = (-4)(3) + (5)(2) + (0)(3) = -2$$

Example – Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

- Element c_{12} is found by summing the products of elements in row 1 of **A** and column 2 of **B**:

$$c_{12} = (-4)(2) + (5)(-4) + (0)(1) = -28$$

Example – Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

- Element c_{21} is found by summing the products of elements in row 2 of **A** and column 1 of **B**:

$$c_{21} = (2)(3) + (0.5)(2) + (3)(0) = 7$$

Example – Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix}$$

- Element c_{22} is found by summing the products of elements in row 2 of **A** and column 2 of **B**:

$$c_{22} = (2)(2) + (0.5)(-4) + (3)(1) = 5$$

Example – Matrix Multiplication

- Solution:

$$\begin{bmatrix} -4 & 5 & 0 \\ 2 & 0.5 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 2 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -28 \\ 7 & 5 \end{bmatrix}$$

Practice Problems

- Find **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 52 \\ 18 \\ 25 \end{bmatrix}$$

Practice Problems

- Find **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 5 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 10 & 25 \\ 4 & 13 & 9 \\ -5 & 10 & 25 \end{bmatrix}$$

Practice Problems

- Find **C** = **AB**

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 2 & -1 \\ 0 & 3 & 5 \end{bmatrix}$$

Note: B is the *identity matrix*

Matrix Multiplication

- In general, matrix multiplication is not commutative:

$$\mathbf{AB} \neq \mathbf{BA}$$

Transpose of a Matrix

- The transpose of a matrix is obtained by switching its row and columns
- The transpose of a matrix is designated by a superscript T:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- The transpose can also be designated with a prime symbol ($\mathbf{A}'$). This is the nomenclature used in MATLAB

Inverse of a Matrix

- Some square matrices have an inverse
- If the inverse of a matrix **A** exists (designated by -1 superscript), then

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

where **I** is the identity matrix – a square matrix with 1's as the diagonal elements and 0's as the other elements

Inverse of a Matrix

- In general, computing the inverse of any matrix of nontrivial size (larger than 3x3, say) is tedious
  - Note: inverse only exists if the determinant is nonzero
- One of the most important matrix operations in MATLAB and Excel is computing the inverse of a (square) matrix
- It is useful in solving systems of linear equations

Inverse of a Matrix

- Example: find inverse of **A**: $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

$$|\mathbf{A}| = (3)(4) - (1)(2) = 10$$

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Check Result

$$\mathbf{A} \times \mathbf{A}^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$