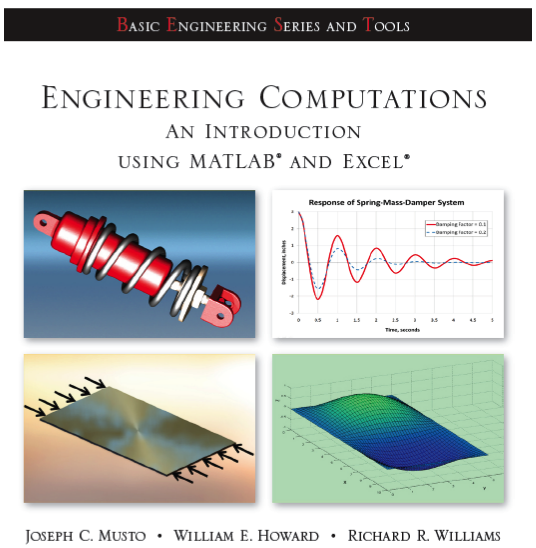
Analytic and Algorithmic Solutions



Chapter 1
Computing Tools

Computing Proficiency for Engineers

- Software Helps Engineers Do Their Jobs
  - **Get Information:** *Web of Science, Google*
  - **Present information and Proposals:** *PowerPoint*
  - **Design:** *Discipline-specific tools, General Computational Tools*
  - **Analyze:** *Computational Tools, Discipline-specific tools*
    - Matlab, Excel, Mathematica, MathCad, Finite Element
  - **Test and Implement:**
    - Data acquisition: LabView, Basic Stamp, etc.
    - Statistics: SPSS, SAS, MiniTab, S+, R, etc.
  - **Report:** *Word, Adobe Design Suite, Framemaker, LaTeX, emacs*

Computational Tools

- Used for "everyday" tasks of engineering:
  - Programming for repetitive calculations
  - Data analysis
  - Plotting
- Many choices:
  - Programming languages such as C++, Fortran, BASIC
  - Mathematical computational tools such as MATLAB, Mathematica, Mathcad, Maple
  - Spreadsheets, such as Microsoft Excel

Computational Tools

- MATLAB
  - Product of MathWorks, Inc.
  - Widely used by engineers in academia and in industry
  - Combines a calculator-type interactive mode and powerful programming tools
- Excel
  - Spreadsheet application of Microsoft Office
  - Originally designed for business use, but now contains many commands and features that are useful for engineering analysis

Analytic and Algorithmic Solutions

- Analytic: Exact solution, based on the application of mathematics
  - Computing tools can help with calculations (like calculators)
- Algorithmic: ~~Approximate~~ Solution based on the application of a computational procedure (the *algorithm*)
  - Algorithm = precisely-defined sequence of steps
  - Instruct the computer to execute a particular algorithm to produce an answer
  - The answer may be exact or approximate, depending on the algorithm

Example

- Consider this problem of projectile motion:

  A ball is fired with an initial speed of 10 m/s, at an angle of 35 degrees relative to the ground. We want to find the maximum height, the location at which the ball hits the ground, and the total flight

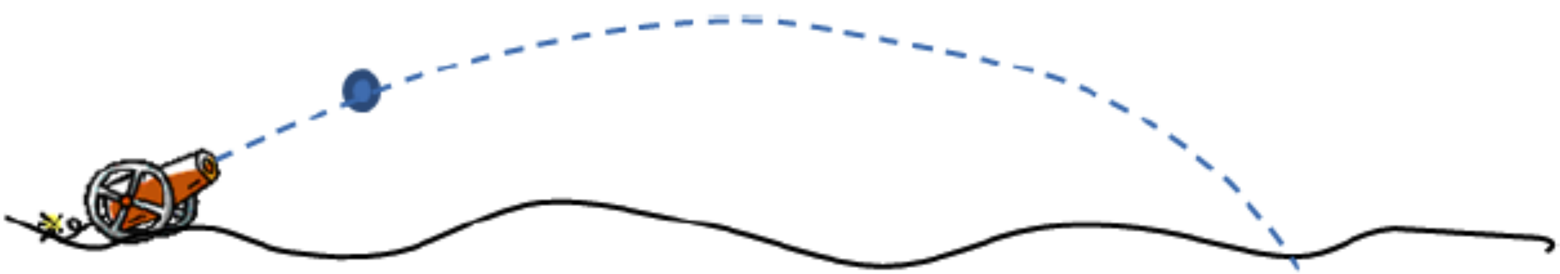


Figure 1.1

The First Step

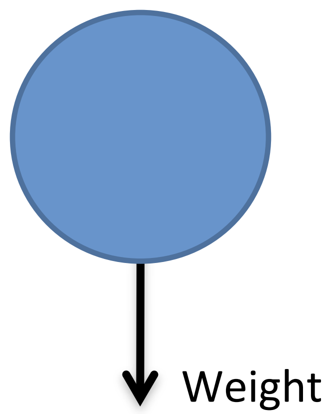
- Set up a model of the system
  - Identify "knowns" and "unknowns" (variables)
  - Understand how they are related (use science – in this case, physics)
- Set up equation(s) relating the variables according to what you know
  - Sometimes there will be many equations
  - Sometimes (in the real world) they will be rather complicated

Two ways to find the answers

- Analytic Solution
  - Use calculus to solve the problem, find the point of maximum height
  - Plug the result in to find the time the flight ends
- Algorithmic Solution
  - Develop an algorithm to evaluate the equation at many time instances
    - Test each function evaluation for meeting prescribed criteria, stop when criteria are satisfied
  - Also called "numerical" solution

The Mathematical Model

- In physics class, you learn how to formulate the mathematical model – the equations describing the physical behavior
- Start by examining the forces acting on the ball:



Mathematical Model (cont.)

- In the y (vertical) direction, the total force is equal to –W (the weight, acting downward)
- The weight is equal to the mass times the acceleration (F = ma)
- Since the weight is equal to mass times g (gravitational acceleration), the vertical acceleration $a_y = -g$

Mathematical Model (cont.)

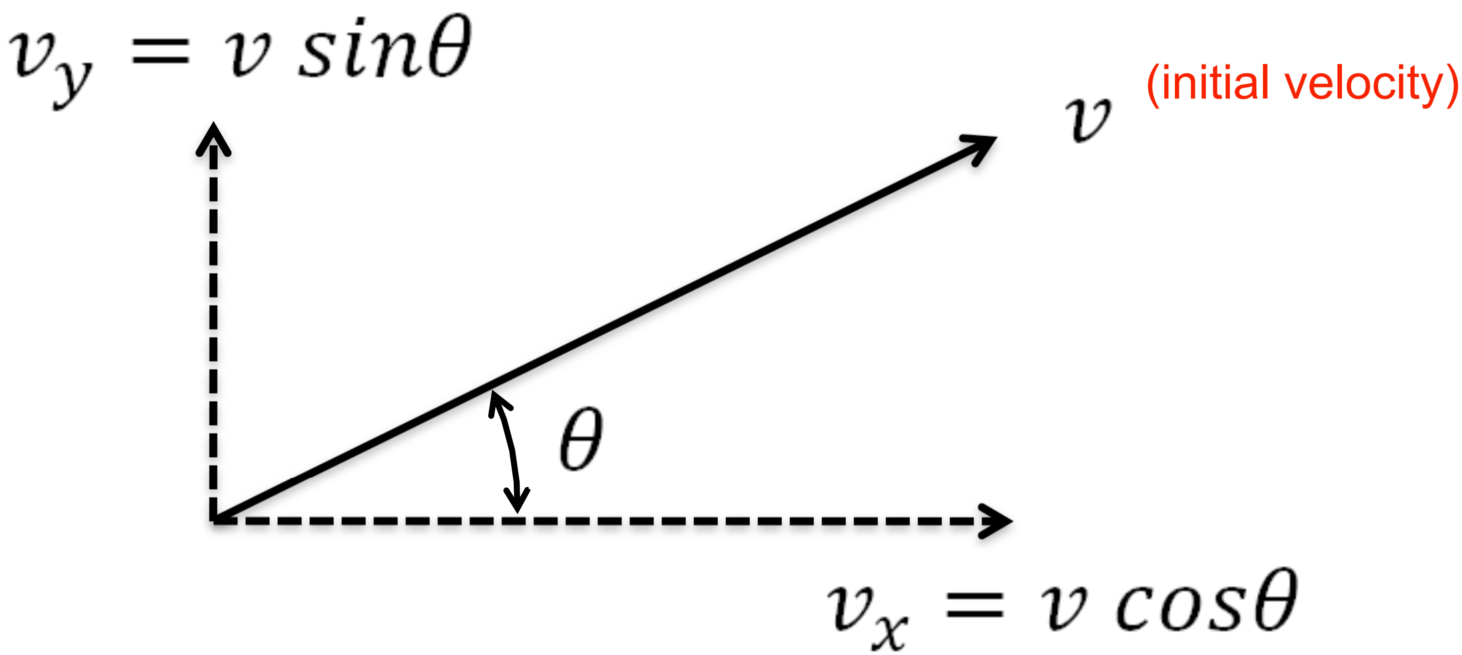
- Once we know the vertical acceleration, we integrate with respect to time to obtain the vertical velocity:

$$v_y = \int a_y \, dt = \int -g \, dt = -gt + C$$

- The constant C is a boundary condition (or initial condition). If we evaluate the velocity equation at time $t = 0$, we see that C must equal the initial upward velocity

Mathematical Model (cont.)

- Initial velocities:



Mathematical Model (cont.)

- So the vertical velocity is:

$$v_y = \; -gt + vsin\theta$$

- Integrating this expression with respect to time gives us the vertical position (height):

$$h \; = \int v_y \, dt = \; \int (-gt + v \; sin\theta) dt = \; -\frac{1}{2} g t^2 + v \; sin\theta \; t + C$$

- Evaluating at time = 0, since the initial height = 0, the constant *C* = 0:

$$h = \; -\frac{1}{2} g t^2 + \; v \; sin\theta t$$

Mathematical Model (cont.)

- Similarly, in the horizontal direction, there are no forces acting on the ball, so

$$a_x = 0$$

- Integrating with respect to time gives us the horizontal velocity:

$$v_x = \int a_x \, dt = \int 0 \, dt = C$$

- When $t = 0$, we see that C is equal to the initial horizontal velocity $v_x = v \cos\theta$

Mathematical Model (cont.)

- Integrating again, we find that the horizontal position x is

$$x = \int v_x \, dt = \int v \, cos\theta \, dt = vcos\theta t + C$$

- When $t = 0$, $x = 0$, since we have defined the origin of our coordinate system at the cannon. Therefore,

$$x = vcos\theta t$$

Mathematical Model (cont.)

- So we now have equations for the height and the horizontal distance as functions of time:

$$h(t) = vtsin\theta - \frac{1}{2}gt^2$$

$$x(t) = vtcos\theta$$

Assumptions

- Before proceeding to the solution of the problem, it is important to recognize the assumptions that have been made in this formulation of the mathematical model:

1. There is no air resistance (the weight is the only force considered)
2. The ground if flat and level (the height is measured relative to the initial position)
3. The launch point is even with the ground (the height of the cannon is neglected; initially h = 0)

Analytic Solution

- This solution requires calculus (as did the formulation of the mathematical model)
- To find an extreme value of the height, we differentiate the expression for height with respect to time:

$$\frac{dh(t)}{dt} = v\ sin\theta - gt$$

- Of course this *rate of change* of height is the vertical velocity, and is equal to zero when an extreme value is reached

Analytic Solution (cont.)

- Setting the rate of change of height equal to zero, we get the value of time for which the height is maximized:

$$t = \frac{v\ sin\theta}{g}$$

- Substituting the values of the constants,

$$t = \frac{\left(10.0\,\frac{m}{s}\right)\sin\left(35.0°\right)}{\left(9.81\,\frac{m}{s^2}\right)} = 0.585\ s$$

Analytic Solution (cont.)

- To find the maximum height, we substitute the time t = 0.585 seconds into the equation for height:

$$h(t = 0.585\ s) = \left(10.0\,\frac{m}{s}\right)(0.585\ s)\sin(35.0°) - \frac{1}{2}\left(9.81\,\frac{m}{s^2}\right)(0.585\ s)^2$$

So h_{max} = 1.68 meters

Analytic Solution (cont.)

- To find the total time of the flight, set the height equal to zero (since the height will be zero when the ball lands):

$$\left(10.0\frac{m}{s}\right)t\sin(35.0°)-\frac{1}{2}\left(9.81\frac{m}{s^2}\right)t^2=0$$

- There are two solutions to this equation: $t = 0$ (the starting point) and $t = 1.17$ seconds (the landing point)

Analytic Solution (cont.)

- To find the location where the ball hits the ground, we substitute the time t = 1.17 seconds into the equation for horizontal position x:

$$x(t = 1.17\ s) = \left(10.0\frac{m}{s}\right)(1.17\ s)\cos(35.0°)$$

So x_{max} = 9.58 meters

Table 1.1 Results of the Analytic Solution

Peak height reached:	1.68 meters
Horizontal distance travelled:	9.58 meters
Total flight time:	1.17 seconds

Aside on Terms

- Your textbook implies that "algorithmic" means "approximate". This is not the case!
- An algorithm is a precise, bounded procedure for solving a problem, e.g.:
  - Sorting a list of records
  - Ranking web pages that match a given string
  - Determining the minimum cost tour of a set of cities
- Many algorithms produce exact answers!
- What your text calls "algorithmic solution" is really a "numerical solution" obtained via an algorithm
  - The basic idea: repeatedly evaluate a function (perturbing the parameters slightly), searching for a particular result

~~Algorithmic~~ Numerical Solution

- For the ~~algorithmic~~ numerical solution, we will evaluate the equations at different values of time until we find the approximate answer
- The algorithm is the procedure we follow to do the computation and determine when we've got the answer
- Let's consider the maximum height, using the equation

$$h(t) = vtsin\theta - \frac{1}{2}gt^2$$

~~Algorithmic~~ Numerical Solution (cont.)

- Here is our solution algorithm for finding the maximum height, written in *pseudocode:*

- Step 1: Start at a time value of t=0 and $h = 0$
- Step 2: Increase time by adding some small value Δt to t (e.g. $t_{new} = t + \Delta t$)
- Step 3: Plug the new value of t_{new} into Equation 1.1 to get a new value of h , which we will call h_{new}
- Step 4: Compare h and h_{new} :
  - If $h < h_{new}$, then the height is still increasing, and the peak has not been reached. Set $t = t_{new}$, $h = h_{new}$, and return to Step 2.
  - If $h > h_{new}$ then the height has started decreasing. This tells us that the ball reached its peak ***somewhere between*** h and h_{new} .
- Step 5: Assume that the maximum height occurs at the height at the start of the interval h.

~~Algorithmic~~ Numerical Solution (cont.)

- How do we determine what value of time interval to use? We make an initial guess, realizing that we may need to repeat the analysis with a different value. Smaller intervals = better accuracy but more calculation steps, larger intervals = less accuracy but fewer steps.
- Solution with time intervals of 0.1 seconds:

Table 1.2 Step-by-Step Solution Algorithm for Finding h_{max}

Loop	t (sec)	t_{new} (sec)	h (m)	h_{new} (m)	Step 4 Decision
1	0	0.1	0	.52	$h < h_{new}$, loop back to Step 2
2	.1	.2	.52	.95	$h < h_{new}$, loop back to Step 2
3	.2	.3	.95	1.28	$h < h_{new}$, loop back to Step 2
4	.3	.4	1.28	1.51	$h < h_{new}$, loop back to Step 2
5	.4	.5	1.51	1.64	$h < h_{new}$, loop back to Step 2
6	.5	.6	1.64	1.68	$h < h_{new}$, loop back to Step 2
7	.6	.7	1.68	1.61	$h > h_{new}$, set $h_{max} = h$. End the algorithm

~~Algorithmic~~ Numerical Solution (cont.)

- Here is the algorithm for finding the total flight time and distance travelled:

- Step 1: Start at a time value of $t = 0$ and $h = 0$
- Step 2: Increase time by adding some small value Δt to t (e.g. $t_{new} = t + \Delta t$)
- Step 3: Plug the new value of t_{new} into Equation 1.1 to get a new value of h , which we will call h_{new}
- Step 4: Check the value of h_{new} :
  - If $h_{new} > 0$, then the cannon ball is still in flight. Set $t = t_{new}$, $h = h_{new}$, and return to Step 2.
  - If $h_{new} < 0$, then the cannon ball hit the ground *somewhere between* h and h_{new} .
- Step 5: Approximate the total flight time by setting $t_{flight} = \frac{t+t_{new}}{2}$.
- Step 6: Find the horizontal distance travelled during the flight by substituting t_{flight} into Equation 1.2. End the algorithm.

~~Algorithmic~~ Numerical Solution (cont.)

- Solution with time intervals of 0.1 seconds:

Table 1.3 Step-by-Step Solution Algorithm for Finding Flight Time and Distance Travelled

Loop	t (sec)	t_{new} (sec)	h_{new} (m)	Step 4 Decision
1	0	0.1	.52	$h_{new} > 0$, loop back to Step 2
2	.1	.2	.95	$h_{new} > 0$, loop back to Step 2
3	.2	.3	1.28	$h_{new} > 0$, loop back to Step 2
4	.3	.4	1.51	$h_{new} > 0$, loop back to Step 2
5	.4	.5	1.64	$h_{new} > 0$, loop back to Step 2
6	.5	.6	1.68	$h_{new} > 0$, loop back to Step 2
7	.6	.7	1.61	$h_{new} > 0$, loop back to Step 2
8	.7	.8	1.45	$h_{new} > 0$, loop back to Step 2
9	.8	.9	1.19	$h_{new} > 0$, loop back to Step 2
10	.9	1.0	0.84	$h_{new} > 0$, loop back to Step 2
11	1.0	1.1	0.38	$h_{new} > 0$, loop back to Step 2
12	1.1	1.2	-0.17	$h_{new} < 0$, set $t_{flight} = \frac{t+t_{new}}{2}$ and find distance using Equation 1.2. End the algorithm.

~~Algorithmic~~ Numerical Solution (cont.)

- Based on the solution for total flight time, calculate the total distance travelled from the equation:

$$x(t) = vtcos\theta$$

- Solution:

Table 1.4 Results of the Algorithmic Solution

Peak height reached:	1.68 meters
Horizontal distance travelled:	9.42 meters
Total flight time:	1.15 seconds

~~Algorithmic~~ Numerical Solution (cont.)

- Results with different values of time intervals:

Time Interval, seconds	Peak Height, meters	Distance Travelled, meters	Total Flight Time, seconds	Number of Loops Calculated
0.50	1.6416	10.2394	1.2500	3
0.25	1.6416	9.2155	1.1250	5
0.10	1.6757	9.4202	1.1500	12
0.05	1.6757	9.6250	1.1750	24
0.01	1.6767	9.5431	1.1650	117
0.001	1.6768	9.5800	1.1695	1170
0.0001	1.6768	9.5788	1.1693	11694

- Compare to exact solution:

1.6768	**9.5789**	**1.1694**

Comparison of Solutions

- Note that the Numerical solution *converges* to the exact solution as the time interval is decreased
- How many intervals should be used? Depends on how accurate the solution needs to be – we will discuss this further when we address accuracy and precision
- For now, note the standard tradeoff:
  - Smaller intervals ➔ more accurate, more computation
  - Larger intervals ➔ less accurate, less computation

Why a Numerical Solution?

- Q: Why use a numerical "search" (as in this example) that produces an approximate answer, if an exact solution exists?
- A1: Engineers often face problems for which an exact solution either
  - Does not exist, or
  - Is impractical to calculate
- A2: A numerical solution can serve as a "sanity check" to verify that an exact solution is correct

A Note About the Analytic Solution

- Remember that what we call the "exact" analytic solution is the exact solution of the mathematical model – *not* of the physical problem
  - In the case of the cannon ball, recall that we made several assumptions, such as neglecting air resistance
- The actual flight of the cannon ball may be very different from the prediction of the mathematical model, depending on the validity of the assumptions

Numerical Solution Using Tools

- Finding the maximum height with Excel and MATLAB