Any problems referring to $A$, $B$ or $C$ refer to the following sets: $A = \{2, 3, 4, 5\}$, $B = \{a, b, c\}$, and $C = \{7, 8, 9, 10, 11\}$.

**Problem 0.** What is the minimum number of cards that can be selected from a standard 52-card deck to guarantee that at least 3 cards of each suit are chosen?

Note this isn’t the problem it looks like it should be. It was supposed to be a Pigeonhole Principle problem, but the way it is stated, the PP doesn’t help. To see this, think about drawing cards out and sorting them into four boxes according to their suit. The problem says that we want each box to have at least three cards in it. This is true if the box (suit) with the minimum number contains three cards.

Now the Refined, Purified Pigeonhole Principle (RPPP) says that the minimum is at most the average. If we draw out $n$ cards, the average number per box will be $n/4$. So this gives us an upper bound on the average, but we want a lower bound (i.e. the minimum is at least 3).

If the problem had said, “What is the minimum number . . . to guarantee that some suit has at least 3 cards?” then we would need to get a lower bound on the maximum number of cards in each box, and since the RPPP says that the maximum is at least the average, it could help us. In that case, by ensuring that the average is 3 (i.e. $n = 12$), we would ensure that the maximum is at least 3.

As the problem is written, in the worst case you draw all 13 cards of one suit, then all 13 of another suit, and then all of a third suit before you finally get three of the fourth suit. Thus, to guarantee that you get three of each suit you have to draw out $3 \times 13 + 3 = 42$ cards.

**Problem 1.** The following 0-1 array represents a relation:

\[
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Is this relation...

a. symmetric? No, not symmetric. For symmetry, the matrix should be equal to its transpose, i.e. $M[i, j] = M[j, i]$ for all $i$ and $j$. That is not true here, e.g. $M[3, 2] = 1$ but $M[2, 3] = 0$.

b. reflexive? Yes. Only 1’s on the main diagonal means it’s reflexive.

c. transitive? No, not transitive, because for example $M[4, 2] = 1$ and $M[2, 1] = 1$ (meaning (4, 2) and (2, 1) are in the relation) but $M[4, 1] = 0$ (meaning (4, 1) is not in the relation. To show that a relation is not transitive we just have to show $i, j, k$ such that ($i, j$) and ($j, k$) are in the relation but ($i, k$) is not. Alternatively, you could do the matrix calculation and show that $M^2 \not\subseteq M$. 


d. antisymmetric? No, not antisymmetric. The matrix condition for antisymmetry is $M \cap M^T \leq I$, where $M^T$ is the transpose of $M$ and $\cap$ is element-wise multiplication. In other words for all $i$ and $j$ such that $i \neq j$, either $M[i, j] = 0$ or $M[j, i] = 0$. That’s not the case for $M[3, 1]$ and $M[1, 3]$, for example.

**Problem 2.** Prove that $n^3 + 2n$ is divisible by three, for any $n \geq 0$.

The proof is by induction on $n$.

$S(n)$ is “$n^3 + 2n$ is divisible by three”.

For the basis, we have to prove $S(0)$, i.e: $0^3 + 2(0)$ is divisible by 3. Obviously this is true because 0 is divisible by any number other than 0.

For the inductive step, we have to show that $S(k) \rightarrow S(k+1)$ for any $k \geq 0$. To prove that, we assume $k \geq 0$ and $S(k)$, i.e. $k^3 + 2k$ is divisible by 3. Now we observe:

\[
(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + (3k^2 + 3k + 3) = (k^3 + 2k) + 3(k^2 + k + 1)
\]

By the inductive hypothesis, the first term in the sum on the last line is divisible by 3, and therefore the whole expression is. Thus $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

Therefore, by mathematical induction, $n^3 + 2n$ is divisible by 3 for any $n \geq 0$.

**Problem 3.** Let $f : A \rightarrow C$ be a function, and let $R \subseteq C \times C$ be a relation on $C$. Write down the definition of each of the following without using words, i.e. using only symbols. (Hint: use quantifiers.)

a. $f$ is one-to-one: $\forall x, y \in A \ [f(x) = f(y) \rightarrow x = y]$.

b. $R$ is antisymmetric: $\forall x, y \in C \ [(x, y) \in R \land (y, x) \in R \rightarrow x = y]$.

c. $f$ is onto: $\forall y \in C \exists x \in A \ [f(x) = y]$.

d. $R$ is transitive: $\forall x, y, z \in C \ [(x, y) \in R \land (y, z) \in R \rightarrow (x, z) \in R]$.

**Problem 4.** Prove that for any integer $n$ greater than 4, $n^2 < 2^n$.

The proof is by (what else) induction on $n$. $S(n)$ is “$n^2 < 2^n$”. The Basis is $S(5)$, i.e. $5^2 < 2^5$, which is true. For the inductive step, the Inductive Hypothesis is that for some $k \geq 4$, $k^2 < 2^k$.

We have to show that $(k + 1)^2 < 2^{k+1}$. We observe that $(k + 1)^2 = k^2 + 2k + 1$, which is less than $2^k + 2k + 1$ by the inductive hypothesis. This is less than $2^{k+1}$ if $2k + 1 < 2^k$, or $k < 2^{k-1} - \frac{1}{2}$. This is clearly true for $k \geq 5$, so that completes the proof.
Problem 5. Write down $C \times B$ and $B \times A$.

\[
C \times B = \{(7, a), (7, b), (7, c), (8, a), (8, b), (8, c), (9, a), (9, b), (9, c), (10, a), (10, b), (10, c), (11, a), (11, b), (11, c)\}
\]
\[
B \times A = \{(a, 2), (a, 3), (a, 4), (a, 5), (b, 2), (b, 3), (b, 4), (b, 5), (c, 2), (c, 3), (c, 4), (c, 5)\}
\]

Problem 6. The relation $R_0$ is defined by: $R_0 = \{(x, y) | x \in A \wedge y \in B \wedge x \mid y\}$, where “$m \mid n$” is read “$m$ divides $n$”, and means that $m$ is a divisor of $n$, i.e. there exists $k$ such that $n = km$. \textbf{Note: this problem doesn’t make sense as stated. It should say: “…y \in C…”}.

a. Write down $R_0$ explicitly (i.e. as a set of pairs).

\[
R_0 = \{(2, 8), (2, 10), (3, 9), (4, 8), (5, 10)\}
\]

b. Write down the matrix representation of $R_0$.

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

c. Draw a directed graph representing $R_0$.

Problem 7. Write down, using explicit set notation, a one-to-one function from $A$ to $C$.

\[
\{(2, 7), (3, 8), (4, 9), (5, 10)\}
\]

(Of course there are lots of other examples.)

Problem 8. Write, using explicit set notation, a function from $C$ to $A$ that is onto.

\[
\{(7, 2), (8, 3), (9, 4), (10, 5)\}
\]

Problem 9. Consider the following relation $R_1 \subseteq A \times A$: $R_1 = \{(x, y) | x^2 < y\}$. Is $R_1$
a. reflexive? No, \( x^2 < x \) is not true for any positive integer.

b. symmetric? No, can’t have have \( x^2 < y \) and \( y^2 < x \) both true.

c. transitive? Yes, since \((2,5)\) is the only pair in the relation, the implication in the definition is vacuously true.

d. antisymmetric? Yes, again since there is only one pair, the implication is vacuously true.

**Problem 10.** Consider the following relation on \( C \):

\[ \{(8,8), (9,10), (10,8), (9,8), (7,7), (9,9)\} \]

Is this relation a partial order? Justify your answer, showing that you know the definition of a partial order and what it means.

To be a partial order, it has to be reflexive, transitive and antisymmetric. It is not reflexive, because \((10,10)\) is not in the relation, so it is not a partial order. (It is, however, transitive and antisymmetric.)

**Problem 11.** Professor Bitdiddle gave a pop quiz with 20 points possible. As he was returning the graded papers, the old Professor observed that “every student got a different score”. Ace Whizkid raised his hand and said “Professor, I think you made a mistake. That’s not possible.” How many students were in the class?

There were at least 22 students in the class. That fact, combined with the fact that there were 21 possible scores (0 through 20), enabled Ace to conclude, using the Pigeonhole Principle, that at least two students had the same grade.

**Problem 12.** Prove that in any group of 100 people there are at least 9 who were born in the same month. (Hint: use the refined, purified pigeonhole principle.)

In a group of 100 people, the average number of people born in each month is \( \frac{100}{12} = 8.333 \). The RPPP says that the maximum is at least the average, so the maximum number of people born in one month is at least 9.