Reading: Read all of Chapter 11 of Grimaldi.

Problem A: Farkel. “Farkel” is a game played with six 6-sided dice. A player starts off rolling all six dice. Certain combinations of numbers score points. For example, any 1 is worth 100 points, and any 5 is worth 50 points. Three of a kind also scores, as do a straight (i.e. six dice showing 1-2-3-4-5-6), four of a kind, five of a kind, six of a kind, or three pair. (The amount of points scored is not important for this problem.) If a roll does not turn up any of these combinations, it is called a “Farkel”, the rolling player’s turn is over, and no points are scored.

Strategy enters the picture because after a player rolls, he puts the dice that “counted” off to the side, and then must choose whether to roll again but without the dice that counted. If you roll again you can increase your score, but you also risk getting a Farkel. For example, if Sue starts out by rolling 1-3-3-5-6-6 initially, she scores 150 points, and must decided whether to continue with only four dice, or stop with 150 points.

It helps to know the probability of “Farkeling” with different numbers of dice. For example, if you roll 2 dice, the only way to score is to get at least one 1 or at least one 5. The “Farkel” rolls are just those in which neither dice lands with a 1 or a 5 facing upward.

When calculating the probability of rolling certain numbers, it often helps to think of the dice as being distinguishable (say, each is a different color). To see why this helps, you need to know that we calculate the probability of a certain “event”—say, at least one die lands with 1 facing up—as a fraction, the numerator of which is the number of ways $n$ dice can land (let’s call these “configurations”) so that the “event” of interest happens (i.e. one or more 1s is facing up), and the denominator of which is the total number of configurations, or ways $n$ dice can land. Distinguishing between the two dice helps in counting the number of configurations in the event of interest. (We are assuming the dice are not loaded, i.e. the dice are all the same and every face is equally likely to land facing up, so that each possible configuration is equally likely.) In this case, there are 6 possible ways for each dice to land, so by the product rule there are a total of $6^n$ different configurations given $n$ distinguishable dice. For example, there are $6^5 = 46,656$ possible configurations with six dice.

Suppose we roll two dice. Clearly 36 configurations are possible. What is the probability of rolling at least one 1 with two dice? Let the two dice be called $a$ and $b$, and let $A_1$ be the set of configurations in which $a$ lands with 1 facing up. (For conciseness from now on we’ll just say “[the value of] $a$ is 1” instead of the more accurate but clumsy “die $a$ lands with 1 facing up”.) Similarly, let $B_1$ be the set of configurations in which $b$ is a 1. It should be easy to see that $|A_1| = 6$, because there are six possibilities for $b$ (and only one for $a$: 1). Similarly $|B_1| = 6$.

Now, $|A_1 \cup B_1| = |A_1| + |B_1| - |A_1 \cap B_1|$ by the Inclusion-Exclusion principle. (The terms in the intersection are counted twice so we have to subtract them.) We know $|A_1 \cap B_1| = 1$ (snake eyes is the only configuration in that set), so we end up with $|A_1 \cup B_1| = 11$, and the probability of rolling...
at least one 1 with two dice is 11/36, or just under 1/3.

a. What is the probability of rolling either a (at least one) 1 or a (at least one) 5, with two dice?
   (Hint 1: Clearly \( |A_5 \cup B_5| = |A_1 \cup B_1| = 1 \), and the size of their intersection is quite easy to compute. Hint 2: The probability is \( |A_5 \cup B_5 \cup A_1 \cup B_1|/36 \).) (Note that the probability of Farkeling is 1 minus this quantity.)

b. What is the total number of possible configurations when three (distinguishable) dice are rolled?

c. What is the probability of getting at least one 1 or at least one 5 when three dice are rolled?
   (Note: it may be easier to calculate the probability of the complement event, i.e. of getting no 1s or 5s with three dice.)

d. How many ways are there to get three-of-a-kind with three dice? (I.e. how many ways for all the dice to land with the same value facing up.)

e. How many ways are there to roll a Farkel (no 1’s, no 5’s, and no 3-of-a-kind) with three dice?

f. What is the probability of Farkeling with three dice?

g. Repeat questions 2–6 for four distinguishable dice.

Graph Theory

Section 11.1: Do problem 14. (You may want to look at Problem 8 in Section 11.2 before doing problem 14.)
Section 11.2: Do problems 6, 8, 10, 12.
Section 11.3: Do problems 2, 4a, 8, 20.
Section 11.4: Do problems 4, 8ab, 10, 12a.